A CORRESPONDENCE BETWEEN HILBERT POLYNOMIALS AND CHERN POLYNOMIALS OVER PROJECTIVE SPACES

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ABSTRACT. We construct a map ζ from $\mathrm{K}_0(\mathbb{P}^d)$ to $(\mathbb{Z}[x]/x^{d+1})^\times \times \mathbb{Z}$, where $(\mathbb{Z}[x]/x^{d+1})^\times$ is a multiplicative Abelian group with identity 1, and show that ζ induces an isomorphism between $\mathrm{K}_0(\mathbb{P}^d)$ and its image. This is inspired by a correspondence between Chern and Hilbert polynomials stated in Eisenbud [1, Exercise 19.18]. The equivalence relation of these two polynomials over $\mathbb{P}^d$ is discussed in this paper.

0. Introduction

It is known that the Hilbert polynomial of a coherent sheaf over a projective scheme is closely related to the Chern polynomial of the sheaf by Hirzebruch-Riemann-Roch theorem. Throughout the paper, $\mathbb{P}^d$ denotes a projective space over an algebraically closed field. In fact, over $\mathbb{P}^d$, knowing the Hilbert polynomial is equivalent to knowing the Chern polynomial. This fact is pointed out in Eisenbud [1, Exercise 19.18]. We will briefly describe these definitions associated with a coherent sheaf in the next section. The Chern and Hilbert polynomials are quite different in terms of degrees and coefficients. The Hirzebruch-Riemann-Roch theorem makes a connection from one to the other. In this paper, we let $\mathcal{A}_0$ and $\mathcal{B}$ denote two Abelian groups which are generated by Chern polynomials and Hilbert polynomials respectively. We prove in Theorem 1 the existence of an isomorphism between the Grothendieck group $\mathrm{K}_0(\mathbb{P}^d)$ and $\mathcal{A}_0 \times \mathbb{Z}$. This appears to be an analogous fact that $\mathrm{K}_0(\mathbb{P}^d)$ and $\mathcal{B}$ are isomorphic as it is shown in [1]. Let $P(t)$ and $C(x)$ denote the Hilbert and Chern polynomials of a coherent sheaf respectively. Our main discussions on the isomorphisms mentioned in the above conclude the following three equivalent statements: for any two coherent sheaves $\mathcal{M}$ and $\mathcal{N}$,

(1) $\mathcal{M}$ and $\mathcal{N}$ represent the same class in $\mathrm{K}_0(\mathbb{P}^d)$,
(2) $P_{\mathcal{M}}(t) = P_{\mathcal{N}}(t)$,
(3) $C_{\mathcal{M}}(x) = C_{\mathcal{N}}(x)$ and $\operatorname{rank} \mathcal{M} = \operatorname{rank} \mathcal{N}$.

It is easy to see (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) by definitions. (2) $\Rightarrow$ (1) is done in [1]. We give a proof for (3) $\Rightarrow$ (1).

The paper is arranged in the following way. Section 1 contains some necessary background materials. It is not possible to give precise definitions in this paper. We state their important properties which are often used in our discussion. Section 2 describes two isomorphic group structures on $\mathrm{K}_0(\mathbb{P}^d)$. One is induced by a map $\eta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{B}$, $\mathcal{B} \subset (\mathbb{Q}[t]/(t^{d+1}))^+$ ([1, Exercises 19.16 and 19.17]) and the other is induced by $\zeta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{A}$, $\mathcal{A} \subset (\mathbb{Z}[x]/(x^{d+1}))^\times \times \mathbb{Z}$ (Theorem 1). The

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above equivalent statements on equivalent classes, $P_{\mathcal{M}}(t)$ and $C_{\mathcal{M}}(x)$ then follow from Theorem 1. The isomorphism from $\mathcal{A}$ to $\mathcal{B}$ induced by ζ^{-1} and η recovers the Hirzebruch-Riemann-Roch theorem. This result together with the fact provided in Eisenbud [1, Exercise 19.18] gives an algorithm which computes the Hilbert polynomial of a coherent sheaf from its Chern polynomial and vise versa without using an explicit sheaf structure. This one-to-one correspondence between the two polynomials is discussed in Section 3.

1. Background

We begin with the definitions of some notations which are used throughout the paper. Let $\mathbb{P}^d$ be a projective space over an algebraically closed field. For any coherent sheaf $\mathcal{M}$ over $\mathbb{P}^d$, $\mathcal{M}$ is associated with a graded finitely generated module $M = \oplus_n M_n$ over the polynomial ring $S = k[x_0, \ldots, x_d]$ with homogeneous components M_n. Conversely, a graded module also defines a coherent sheaf, but there are usually more than one graded module associated to a given coherent sheaf (*c.f.* [5, 7]). For each $\mathcal{M}$, there exists the *Hilbert polynomial* $P_{\mathcal{M}}(t)$ of $\mathcal{M}$ such that for any large enough integer $n \in \mathbb{N}$, $P_{\mathcal{M}}(n)$ coincides the value of the Hilbert function of the module M, $H_M(n) = \operatorname{length}(M_n)$. For example, any twisted structure sheaf $\mathcal{O}(-m)$ with $m \in \mathbb{Z}$ is associated with the graded module $S[-m]$ and $P_{\mathcal{O}(-m)}(t) = \binom{t+d-m}{d}$. The Hilbert polynomials are additive on short exact sequences of sheaves.

The *Chern polynomial* $C_{\mathcal{M}}(x)$ (often called the *total Chern class*) of $\mathcal{M}$ is a formal sum of the Chern classes which, in geometry, are usually viewed as cycles in the cohomology groups (*c.f.* Griffiths and Harris [4]) or operators on the Chow groups (*c.f.* Fulton [2] or Roberts [8]). The Chern classes considered in this paper are in the latter form. In general, their definitions are very complicated. We describe these notions for sheaves over $\mathbb{P}^d$ and recall a few properties that will be useful for the latter discussions. The complete details can be found in the above cited references.

The *Chow group* $\mathrm{A}_*(\mathbb{P}^d)$ of $\mathbb{P}^d$ is generated by the linear subspaces $\mathbb{P}^{d-\ell}, \ell = 0, \ldots, d$, so it has a simple structure; $\mathrm{A}_*(\mathbb{P}^d) \cong \mathbb{Z}^{d+1}$. Therefore, the Chern classes can be identified with integers. For a locally free sheaf $\mathcal{M}$ of finite rank r, there exist r Chern classes $c_1(\mathcal{M}), \ldots, c_r(M)$, and the *Chern polynomial* of $\mathcal{M}$ is defined to be a formal sum of $c_i(\mathcal{M})$,

$$C_{\mathcal{M}}(x) = 1 + \sum_{i=1}^{r} c_i(\mathcal{M})x^i = 1 + c_1(\mathcal{M})x + \cdots + c_r(\mathcal{M})x^r \pmod{x^{d+1}}.$$

For instance, $c_1(\mathcal{O}(-m)) = -m$ for any twisted sheaf with $m \in \mathbb{Z}$ and there is no higher Chern classes. Thus, $C_{\mathcal{O}(-m)}(x) = 1 - mx$. An important property called the *Whitney sum formula* states that the Chern polynomials are multiplicative on short exact sequences of sheaves.

Over a nonsingular variety, every coherent sheaf admits a unique minimal resolution of locally free sheaves up to quasi-isomorphisms

$$(1) \quad 0 \to \oplus_{j_d}\mathcal{O}(-j_d)^{\beta_{d,j_d}} \to \cdots \to \oplus_{j_1}\mathcal{O}(-j_1)^{\beta_{1,j_1}} \to \oplus_{j_0}\mathcal{O}(-j_0)^{\beta_{0,j_0}} \to \mathcal{M} \to 0.$$

Using the Whitney sum formula, the definition of Chern classes can be extended to the coherent sheaves. By (1), the Chern polynomial of $\mathcal{M}$ over $\mathbb{P}^d$ is a polynomial

modulo x^{d+1} with integer coefficients

$$C_M(x) = \frac{\prod_{i:\text{even}} \prod_{j_i} (1 - j_i x)^{\beta_{i,j_i}}}{\prod_{i:\text{odd}} \prod_{j_i} (1 - j_i x)^{\beta_{i,j_i}}} \pmod{x^{d+1}} \tag{2}$$

and the Hilbert polynomial is of degree at most d with rational number coefficients

$$P_M(t) \;=\; \textstyle\sum_{i,j_i} (-1)^{\beta_{i,j_i}} P_{S(-j_i)}(t).$$

Recall that $C_{\mathcal{O}(-m)}(x) = 1 - mx$ and $P_{\mathcal{O}(-m)}(t) = \binom{t+d-m}{d}$. The Chern polynomials of locally free sheaves always have integer coefficients and the degree varies while the Hilbert polynomials of such sheaves have rational number coefficients and the degree is fixed by $\dim X = d$. A consequence of the Hirzebruch-Riemann-Roch theorem shows the connection of the Euler characteristic and Chern characters of coherent sheaves which leads a representation of Hilbert polynomial in terms of Chern classes in some special cases. We will recall this in Section 3.

Next, we define the Grothendieck group. The Grothendieck group of locally free sheaves, denoted $\mathrm{K}_0(\mathbb{P}^d)$, is the Abelian group generated by all the locally free sheaves $[\mathcal{M}]$ modulo the subgroup generated by $[\mathcal{M}] - [\mathcal{M}'] - [\mathcal{M}'']$ whenever $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ forms an exact sequence of sheaves. We also denote $[\mathcal{M}]$ by $[M]$ if the sheaf $\mathcal{M}$ is associated with the module M. The Grothendieck group of coherent sheaves $\mathrm{G}_0(\mathbb{P}^d)$ is defined in a similar way. Since every coherent sheaf admits a locally free resolution in the form of (1), $\mathrm{K}_0(\mathbb{P}^d)$ is isomorphic to $\mathrm{G}_0(\mathbb{P}^d)$. Henceforth, we use $\mathrm{K}_0(\mathbb{P}^d)$ and refer it as the Grothendieck group of $\mathbb{P}^d$ for simplicity. The generators of $\mathrm{K}_0(\mathbb{P}^d)$ can be described precisely in the followings. The module $S/(x_0, \ldots, x_d)$ defines a zero sheaf since $(x_0, \ldots, x_d)$ is an irrelevant ideal. We take the Koszul resolution of $S/(x_0, \ldots, x_d)$ and obtain a long exact sequence on locally free sheaves,

$$0 \to \mathcal{O}(-d-1) \to (\mathcal{O}(-d))^{\binom{d+1}{d}} \to \cdots \to (\mathcal{O}(-1))^{\binom{d+1}{1}} \to \mathcal{O} \to 0. \tag{3}$$

This gives a relation for the twisted sheaves in $\mathrm{K}_0(\mathbb{P}^d)$ which expresses $[\mathcal{O}(-d-1)]$ as an alternating sum of $[\mathcal{O}], [\mathcal{O}(-1)], \ldots, [\mathcal{O}(-d)]$. If we twist the exact sequence (3) by a degree, say by degree one, then we have the following exact sequence

$$0 \to \mathcal{O}(-d) \to (\mathcal{O}(-d+1))^{\binom{d+1}{d}} \to \cdots \to (\mathcal{O})^{\binom{d+1}{1}} \to \mathcal{O}(1) \to 0.$$

Thus, $[\mathcal{O}(1)]$ is also generated by the same set of twisted sheaves and similarly for other degrees. This implies that $\mathrm{K}_0(\mathbb{P}^d)$ is generated by $[\mathcal{O}(-m)]$ with $m = 0, \ldots, d$. Furthermore, these are free generators. A brief argument for this fact will be developed in the next section. On the other hand, let S_ℓ denote the graded module of S modulo ℓ variables

$$S_\ell = k[x_0, \cdots, x_d]/(x_{d-\ell+1}, \cdots, x_d).$$

The Koszul complex on $x_{d-\ell+1}, \ldots, x_d$ provides a resolution of locally free sheaves for S_ℓ. By a standard argument, $[S_\ell]$, $\ell = 0, \ldots, d$, also generates $\mathrm{K}_0(\mathbb{P}^d)$.

It is known that different sheaves may have the same Hilbert polynomials and Chern polynomials. However, both polynomials are well-defined for the equivalence

classes of coherent sheaves in the Grothendieck group. We have already seen in Eisenbud [1] that the Hilbert polynomials characterize the classes in $\mathrm{K}_0(\mathbb{P}^d)$. The main goal of this paper is to show that the Chern polynomials do the same job; namely, distinct classes have different pairs of Chern polynomial and rank.

2. Groups Isomorphic to $\mathrm{K}_0(\mathbb{P}^d)$

If a polynomial with rational coefficients has integral values at large integers, then it can be written as a linear combination over $\mathbb{Z}$ of the following binomial coefficient functions in t

$$\binom{t}{0}, \binom{t}{1}, \binom{t}{2}, \ldots, \binom{t}{\ell}, \ldots.$$

These polynomials can be replaced by

$$\binom{t}{0}, \binom{t+1}{1}, \binom{t+2}{2}, \ldots, \binom{t+\ell}{\ell}, \ldots.$$

If we let $a_\ell = \binom{t}{\ell}$ and $b_\ell = \binom{t+\ell}{\ell}$, then $b_\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} a_i$ and $a_\ell = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} b_i$.

Let $\mathbb{P}^d = \mathrm{Proj}(S) = \mathrm{Proj}(k[x_0, \cdots, x_d])$ be as in the previous section. Then, $\binom{t+d}{d}$ is exactly the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^d}$ and $P_{s_\ell}(t) = \binom{t+d-\ell}{d-\ell}$. Since Hilbert polynomials have integral values at large integers, for any graded module M, $P_M(t)$ can be written as a linear combination of $P_{S_\ell}(t), \ell \in \mathbb{N} \cup \{0\}$. Let $\mathcal{B}$ denote the Abelian group generated by all Hilbert polynomials of coherent sheaves over $\mathbb{P}^d$. The group $\mathcal{B}$ is a subgroup of the additive group $(\mathbb{Q}[t]/t^{d+1})^+$ with identity 0. Then, $P_{S_\ell}(t)$, $\ell = 0, \ldots, d$, form a set of generators for $\mathcal{B}$. Moreover, these generators are linearly independent since $\deg P_{S_\ell}(t) = \ell$. Thus, $\mathcal{B}$ is a free Abelian group of rank $d+1$.

Let α be a class in $\mathrm{K}_0(\mathbb{P}^d)$ represented by some sheaf $\mathcal{M}$. The map

$$\eta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{B}$$

takes α to the Hilbert polynomial $P_{\mathcal{M}}(t)$ of $\mathcal{M}$ induces an isomorphism. To see this, we note that η is surjective since P_{S_ℓ}, $\ell = 0, \ldots, d$, generate $\mathcal{B}$. That these generators are linearly independent implies $[S_\ell]$ are also linearly independent. Therefore, $\mathrm{K}_0(\mathbb{P}^d)$ is generated freely by $[S_\ell]$, $\ell = 0, \ldots, d$, and the injectivity follows (*cf.* [1, Exercise 19.17]). Another proof, using $\{[\mathcal{O}(-m)] : m = 0, \ldots, d\}$ as a generating set and in which η is induced by a map taking $[\mathcal{O}(-m)]$ to its Hilbert series, can be found also in [1, Exercise 19.16] in great details.

Alternatively, let $\mathcal{A}_0$ denote the Abelian group generated by all the Chern polynomials of the coherent sheaves over $\mathbb{P}^d$. Similar to $\mathcal{B}$, $\mathcal{A}_0$ is a subgroup of the Abelian multiplicative group $(\mathbb{Z}[x]/x^{d+1})^\times$ with identity 1. Let $\mathcal{A}$ denote the subgroup $\mathcal{A}_0 \times \mathbb{Z}$ of $(\mathbb{Z}[t]/t^{d+1})^\times \times \mathbb{Z}$ which has the natural group structure that for any two elements $(f(x), r)$ and $(g(x), s)$, $(f(x), r) + (g(x), s) = (f(x)g(x), r+s)$ and $(1, 0)$ is the identity. For any α in $\mathrm{K}_0(\mathbb{P}^d)$ represented by a locally free sheaf $\mathcal{M}$, we define a map from $\mathrm{K}_0(\mathbb{P}^d)$ to $\mathcal{A} = \mathcal{A}_0 \times \mathbb{Z}$,

$$\begin{array}{rccc} \zeta : & \mathrm{K}_0(\mathbb{P}^d) & \longrightarrow & \mathcal{A} = \mathcal{A}_0 \times \mathbb{Z} \\ & \alpha = [\mathcal{M}] & \longrightarrow & (C_{\mathcal{M}}(x), \operatorname{rank} \mathcal{M}). \end{array}$$

The map ζ is a well-defined group homomorphism by the Whitney sum formula. It should be noted that the component $\mathbb{Z}$ in $\mathcal{A}$ is necessary in order to distinguish

different classes which have the same Chern polynomial. The simplest examples are $\alpha = [\mathcal{O}_{\mathbb{P}^d}]$ of rank one and $\beta = [\oplus_r \mathcal{O}_{\mathbb{P}^d}]$ of rank $r \neq 0, 1$. Both α and β have the Chern polynomial equal to 1 while $\beta = r\alpha \neq \alpha$ in $\mathrm{K}_0(\mathbb{P}^d)$. Analogous to the isomorphism defined by η, we prove that ζ is also an isomorphism.

Theorem 1. *$\zeta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{A}$ is an isomorphism of Abelian groups.*

The following Lemma 1 implies that $\mathcal{A}$ is free of rank $d+1$. The generators $(1-\ell x, 1)$ of $\mathcal{A}$ are the image of $[\mathcal{O}(-\ell)]$ for all ℓ so η is surjective and therefore, it is an isomorphism since both groups are free of the same rank.

Lemma 1. *The group $\mathcal{A}_0$ is freely generated by $1-x, \dots, 1-dx$. Furthermore, $(1,1), (1-x,1), \dots, (1-dx,1)$ are free generators for $\mathcal{A}$.*

Proof. It is clear that $\mathcal{A}_0$ is generated by $1-x, \dots, 1-dx$ and that $\mathcal{A}$ is generated by $(1,1), (1-x,1), \dots, (1-dx,1)$ by the resolutions (1) and (3). If

$$r_0(1,1) + r_1(1-x,1) + \cdots + r_d(1-dx,1) = (1,0)$$

in $\mathcal{A}$ for some $r_0, \dots, r_d \in \mathbb{Z}$. Then, the following (4) and (5) hold,

$$(1-x)^{r_1} \cdots (1-dx)^{r_d} \equiv 1 \pmod{x^{d+1}}, \tag{4}$$

$$r_0 + r_1 + \cdots + r_d = 0. \tag{5}$$

It suffices to show that $1-x, \dots, 1-dx$ are linearly independent; that is, (4) implies $r_1 = \cdots = r_d = 0$. Then, the linearly independence of $(1,1), (1-x,1), \dots, (1-dx,1)$ follows from (5).

Without loss of generality, we do the following argument assuming that none of $r_1, \dots, r_d$ is zero. (If any of $r_0, \dots, r_d$ is zero, then it follows a similar argument which lead to the same contradiction.) We take the derivative of the equation in (4) and obtain

$$(1-x)^{r_1}(1-2x)^{r_2}\cdots(1-dx)^{r_d}\left(\frac{-r_1}{1-x} + \frac{-r_2}{1-2x} + \cdots + \frac{-r_d}{1-dx}\right) \equiv 0 (\operatorname{mod} x^d).$$

The above product is taken in the unique factorization domain $\mathbb{Z}[[x]]$ and a simple computation shows that

$$(1-x)^{r_1}(1-2x)^{r_2}\cdots(1-dx)^{r_d} = 1 - (r_1 + 2r_2 + \cdots + dr_d)x + \cdots \not\equiv 0 (\operatorname{mod} x^d).$$

Therefore,

$$\frac{r_1}{1-x} + \frac{2r_2}{1-2x} + \cdots + \frac{dr_d}{1-dx} \equiv 0 (\operatorname{mod} x^d).$$

Using the Taylor expansion, we have

$$\begin{aligned}(r_1 + 2r_2 + \cdots + dr_d) + (r_1 + 2^2 r_2 + \cdots + d^2 r_d)x + \cdots \\ +(r_1 + 2^d r_2 + \cdots + d^d r_d)x^{d-1} \equiv 0 (\operatorname{mod} x^d).\end{aligned} \tag{6}$$

The equivalence given by Equation (6) provides a linear system in $r_1, \dots, r_d$ with Vandemonde coefficients if $r_1, \dots, r_d$ are all nonzero. This is a contradiction because a Vandemonde system has only trivial solutions. Therefore, $r_1 = r_2 = \cdots = r_d = 0$ and $r_0 = 0$ by (5). This complete the proof of both assertions in the lemma. ∎

The isomorphisms η and ζ shows that the three groups $\mathrm{K}_0(\mathbb{P}^d)$, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and the following conditions.

Theorem 2. *For any coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ on $\mathbb{P}^d$, the followings are equivalent:*
(1) $[\mathcal{M}] = [\mathcal{N}]$ *in* $\mathrm{K}_0(\mathbb{P}^d)$.
(2) $P_{\mathcal{M}}(t) = P_{\mathcal{N}}(t)$.
(3) $C_{\mathcal{M}}(x) = C_{\mathcal{N}}(x)$ *and* $\operatorname{rank}\mathcal{M} = \operatorname{rank}\mathcal{N}$.

3. The equivalence of Chern and Hilbert polynomials

This section discusses the close relationship of the Chern and Hilbert polynomials which inspires the work presented in the previous section. The Hirzebruch-Riemann-Roch theorem relates the Euler characteristic with the Chern characters. Not much is known about representing the Hilbert polynomial in terms of Chern classes in general. We will discuss this and its converse over $\mathbb{P}^d$ in the current section. Proposition 1 provides a one-to-one correspondence between the two polynomials and an algorithm for a computational purpose.

The Hirzebruch-Riemann-Roch theorem proves that there exists a certain maps from the Grothendieck group of a scheme X to its Chow group and that it commutes with the maps induced by a projective map from X to a point. This theorem for projective spaces induces an expression of the Hilbert function in terms of the Chern classes. In order to make it precise, we need to introduce the Chern characters of $\mathcal{M}$. Suppose the Chern polynomial can be decomposed into

$$C_{\mathcal{M}}(x) = (1-\alpha_1 x)\cdots(1-\alpha_d x) \tag{7}$$

in $(z[x]/(x^{d+1})^{d+1})$. In this case, $\alpha_1, \ldots, \alpha_d$ are called *Chern roots*. Then, the *Chern character* of $\mathcal{M}$ is a power series defined by

$$\operatorname{ch}(\mathcal{M}) = e^{\alpha_1 x} + \cdots + e^{\alpha_d x}. \tag{8}$$

The coefficient of the x^i in the Taylor expansion of (8) is called the *i-th Chern character* of $\mathcal{M}$, denoted $\operatorname{ch}_i(\mathcal{M})$. Since each $\operatorname{ch}_i(\mathcal{M})$ is a symmetric function of α_i and the Chern classes are elementary symmetric functions in α_i, the Chern characters $\operatorname{ch}_i(\mathcal{M})$ can be expressed as a polynomial in the Chern classes. The first few terms are

$$\begin{aligned} \operatorname{ch}(\mathcal{M}) &= r + c_1 x + \tfrac{1}{2!}(c_1^2 - 2c_2)x^2 + \tfrac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3)x^3 + \\ &\quad \tfrac{1}{4!}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4)x^4 + \cdots, \end{aligned} \tag{9}$$

where $c_i = c_i(\mathcal{M})$ and $r = \operatorname{rank}\mathcal{M}$ (see [2, Example 15.1.2] for the exact formulations). We should note that the factorization (7) does not always exit over the current projective scheme. However, the expressions in (9) are independent from the existence of the Chern roots.

For any power series $s(x)$ in x, let $\Phi(s(x))$ denote the *coefficient of x^d in the Taylor expansion of the expression $s(x)\left(\dfrac{x}{1-e^{-x}}\right)^{d+1}$*. Theorem 3 states the Hirzebruch Riemann-Roch theorem for $\mathbb{P}^d$. The details can be found in the references by Fulton and Lang ([2, Example 15.1.4], [3]) and Hirzebruch ([6, Lemma 1.7.1]).

Theorem 3 (Hirzebruch-Riemann-Roch)**.** *Let $X = \mathbb{P}^d$. Then, for any locally free sheaf $\mathcal{M}$ on X,*

$$\Phi(\operatorname{ch}(\mathcal{M})) = \chi(\mathcal{M}), \tag{10}$$

where $\chi(\mathcal{M}) = \sum_{i\geq 0}(-1)^i \dim_k H^i(X, \mathcal{M})$, the alternating sum of the cohomology groups, is the Euler characteristic of $\mathcal{M}$.

Let $\mathcal{M}$ be defined by a finitely generated graded module $M = \oplus M_n$. By the induction on dimension of the support of $\mathcal{M}$, it shows that $H^i(X, \mathcal{M}(n)) = 0$ for all $i > 0$ and

$$\chi(\mathcal{M}(n)) = \sum_{i\geq 0}(-1)^i \dim H^i(X, \mathcal{M}(n)) = \dim H^0(X, \mathcal{M}(n)) = \dim M_n,$$

for $n \gg 0$ (*cf.* Hartshorne [5, Chapter III]). In the case where $\mathcal{M}$ is locally free, we have

$$\Phi(\operatorname{ch}(\mathcal{M}(n))) = \dim H^0(X, \mathcal{M}(n)) = \dim M_n \tag{11}$$

by the Hirzebruch-Riemann-Roch theorem. In particular, $\operatorname{ch}(\mathcal{M}(n)) = \operatorname{ch}(\mathcal{M} \otimes \mathcal{O}(n)) = \operatorname{ch}(\mathcal{M})\operatorname{ch}(\mathcal{O}(n)) = e^{nx}\operatorname{ch}(\mathcal{M})$. We replace n in (11) by an intermediate t. The left hand side of (11), $\Phi(e^{tx}\operatorname{ch}(\mathcal{M}))$, becomes a polynomial in t whose values at large integers agree with values of the Hilbert function of M. Therefore, it is the Hilbert polynomial (*cf.* Fulton [2, Example 15.2.7(a)]).

If $C_{\mathcal{M}}(x) = 1 + c_1(\mathcal{M})x + \cdots + c_r(\mathcal{M})x^r$ is the Chern polynomial of some locally free sheaf $\mathcal{M}$ of rank r, then $\operatorname{ch}(\mathcal{M})$ is known explicitly as it is shown in (9). The Hilbert polynomial $P_{\mathcal{M}}(t)$ of $\mathcal{M}$ obtained by $\Phi(e^{tx}\operatorname{ch}(\mathcal{M}))$ is the coefficient of x^d in $e^{tx}\operatorname{ch}(\mathcal{M})\left(\frac{x}{1-e^{-x}}\right)^{d+1}$. Part A of the following proposition is an immediate consequence of Theorem 3 as it is explained in the above.

Proposition 1. *Let $X = \mathbb{P}^d$ and let $\mathcal{M}$ be a coherent sheaf on X of rank r.*

A. *Let $C_{\mathcal{M}}(x)$ be the Chern polynomial of $\mathcal{M}$. Then the Hilbert polynomial of $\mathcal{M}$ is*

$$P_{\mathcal{M}}(t) = \Phi(e^{tx}\operatorname{ch}(\mathcal{M})). \tag{12}$$

B. *If* $P_{\mathcal{M}}(t) = \sum_{\ell=0}^{d} a_\ell \binom{t+d-\ell}{d-\ell}$, *then*

$$C_{\mathcal{M}}(x) \equiv \prod_{\ell=0}^{d}[C_{S_\ell}(x)]^{a_\ell}, \quad (\text{mod } x^{d+1}), \tag{13}$$

where $C_{S_\ell}(x)$ is the Chern polynomial of S_ℓ as a module over S.

Proof. It remains to prove Part B. We recall from Section 1 that S_ℓ defines the linear subspace $\mathbb{P}^{d-\ell}$ in $\mathbb{P}^d$. Since the Hilbert polynomial of S_ℓ is $\binom{t+d-\ell}{d-\ell}$, by the assumption and Theorem 2, $P_{\mathcal{M}}(t) = \sum_{\ell=0}^{d} a_\ell \binom{t+d-\ell}{d-\ell} = \sum_{\ell=0}^{d} a_\ell P_{S_\ell}(t)$ in $\mathcal{B}$ if and only if $[\mathcal{M}] = \sum_{\ell=0}^{d} a_\ell [S_\ell]$ in $\mathrm{K}_0(\mathbb{P}^d)$. Therefore, $C_{\mathcal{M}}(x) \equiv \prod_{\ell=0}^{d}(C_{S_\ell}(x))^{a_\ell}$ (mod x^{d+1}). $\square$

We note that the expression of $P_{\mathcal{M}}(t)$ in the hypothesis of Part B is always possible since $\binom{t+d-\ell}{d-\ell}, \ell = 0, \ldots, d$, form a basis for the group $\mathcal{B}$ of all the Hilbert polynomials. The correspondence between the two polynomials can be viewed explicitly in the followings. Let σ denote the group homomorphism $\eta \circ \zeta^{-1}$

$$\sigma = \eta \circ \zeta^{-1} : \mathcal{A} \longrightarrow \mathcal{B}.$$

Any element in the group $\mathcal{A}$ of the Chern polynomials can be written as $(\prod_{m=1}^{d}(1-mx)^{r_m}, s)$ for some r_m and s in $\mathbb{Z}$. It is not hard to see that such an element has a preimage via ζ^{-1} in $\mathrm{K}_0(\mathbb{P}^d)$ as $\sum_{m=1}^{d} a_m[\mathcal{O}(-m)] + (s-r)[\mathcal{O}]$ where $a = a_1 + \cdots + a_m$. This implies

$$\begin{aligned} \sigma((\prod(1-mx)^{a_m}, s)) &= \textstyle\sum_{m=1}^{d} a_m \binom{t+d-m}{d} + (s-a)\binom{t+d}{d} \\ &= \textstyle\sum_{m=1}^{d} a_m P_{\mathcal{O}(-m)}(t) + (s-a)P_{\mathcal{O}}(t). \end{aligned}$$

An element $(f(x), s)$ in $\mathcal{A}$ is said to be *representative by* $\mathcal{M}$ (or $\mathcal{M}$ *represents* $(f(x), s)$) if there exists a sheaf $\mathcal{M}$ such that $f(x)$ is the Chern polynomial of $\mathcal{M}$ and $\operatorname{rank} \mathcal{M} = s$. Thus, for any representative $(f(x), s)$ in $\mathcal{A}$, σ takes $(f(x), s)$ to the Hilbert polynomial of M. The computation can be carried out by (12). Conversely, the preimage of the Hilbert polynomial of $\mathcal{M}$ is the pair of the Chern polynomial and the rank of $\mathcal{M}$. This preimage is uniquely determined since $\sigma = \eta \circ \zeta^{-1}$ is an isomorphism. More precisely, (13) computes the Chern polynomial and $a_0 + \cdots + a_d$ indicates the rank which is independent from the choices of a representing sheaf.

We end the paper with the following two remarks which are often considered in the study of this course.

Remark 1. We would like to point out a special case where $\mathcal{M}$ is a twisted structure sheaf $\mathcal{O}(-m)$. This has drawn the attention of those who attempted to solve Exercise 19.18 in [1]. For any $m \in \mathbb{Z}$, we obtained

$$P_{\mathcal{O}(-m)}(t) = \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} P_{S_\ell}(t) \tag{14}$$

by an inductive computation on binomial coefficient functions

$$\binom{t+d-m}{d} = \binom{t+d-(m-1)}{d} - \binom{t+(d-1)-(m-1)}{d-1}. \tag{15}$$

We use the convention that $\binom{a}{b} = 0$ if $a < b$. Part B in Proposition 1 can be reduced to a problem asking the following congruence

$$C_{\mathcal{O}(-m)}(x) = 1 - mx \equiv \prod_{\ell=0}^{m} (C_{S_\ell})^{(-1)^{\ell}\binom{m}{\ell}}, \quad (\mathrm{mod}\ x^{d+1}). \tag{16}$$

Since C_{S_ℓ} can be computed by the Koszul complex as it is shown in (2), a naive attempt on proving the congruence in the above (16) is to compute the coefficients of each x^i on the right hand side and to show that the coefficients of higher terms are zero. However, the coefficient of a general term x^i in terms of the binomial coefficients is rather complicated. It is not clear how these terms vanish for $i \geq 2$ if they are treated as combinatorial formulae.

The result in the previous section provides the following intuition from a different perspective: (16) follows from the fact that σ is a group isomorphism; precisely,

$$\begin{aligned}(C_{\mathcal{O}(-m)}(x),1)=(1-mx,1) &= \sigma^{-1}(P_{\mathcal{O}(-m)}(t))=\sigma^{-1}(\textstyle\sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}P_{S_\ell}(t))\\ &= (\textstyle\prod_{\ell=0}^{m}(C_{S_\ell}(x))^{(-1)^{\ell}\binom{m}{\ell}},1).\end{aligned}$$

Hence, $C_{\mathcal{O}(-m)}(x)$ and $\prod_{\ell=0}^{m}(C_{S_\ell}(x))^{(-1)^{\ell}\binom{m}{\ell}}$ are equal in the groups $\mathcal{A}_0$. Although (15) is a combinatorial property, it is also the relations of Hilbert polynomials of the sheaves in the following short exact sequence

$$0\longrightarrow\mathcal{O}(-m)\xrightarrow{\mathcal{H}}\mathcal{O}(-m+1)\longrightarrow\mathcal{O}_{\mathcal{H}}(-m+1)\longrightarrow 0, \tag{17}$$

where $\mathcal{H}$ is a hyperplane in $\mathbb{P}^d$. Since the Chern polynomials depend on the ambient scheme, a similar inductive decomposition as it is for $P_{\mathcal{M}}(t)$ in (14) does not hold for Chern polynomials. However, (17) induces an identity on the cycles $\alpha=[\mathcal{O}(-m)]=\sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}[S_\ell]$ in $\mathrm{K}_0(\mathbb{P}^d)$ which implies the corresponding (16) in $\mathcal{A}$.

Remark 2. A more fundamental correspondence of the Chern and Hilbert polynomials should be pointed out. Let a_i denote the coefficient of x^i in the Taylor expansion of $(\frac{x}{1-e^{-x}})^{d+1}$. (13) can be written explicitly as

$$\begin{aligned}P_M(t)= &\ \tfrac{1}{d!}a_0rt^d+\tfrac{1}{(d-1)!}(a_0\,\mathrm{ch}_1+a_1r)t^{d-1}+\\ &\cdots(a_0\,\mathrm{ch}_d+a_1\,\mathrm{ch}_{d-1}+\cdots+a_{d-1}\,\mathrm{ch}_1+a_dr),\end{aligned} \tag{18}$$

in which we abbreviate $\mathrm{ch}_i(\mathcal{M})$ by ch_i. Replacing the above ch_i by the proper terms in (9), the coefficients of $P_{\mathcal{M}}$ can be expressed in terms of Chern classes. Conversely, if $P_{\mathcal{M}}$ is known; that is, the coefficients of $P_{\mathcal{M}}(t)$ are determined, then the Chern classes can be solved inductively using the above (18). However, the computation for a_i is tedious. Part B in Proposition 1 avoids such lengthy computation.

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